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That the roots of (1) may be real and separated by those of (2), it is necessary and sufficient that

(*a*) the roots of $g(x)$ shall be real and distinct (such as $\beta_1 < \beta_2 < \cdots < \beta_n$) and

(*b*) $f(-\infty), f(\beta_1), \cdots, f(\beta_n), f(\infty)$ shall have alternate signs.

Now $f(\beta_i) = -R_1(\beta_i)$, and $R_1(x)$ cannot change sign more than $n-1$ times. Hence, (*a*) and (*b*) are equivalent to the conditions that c_0 shall be positive, and that the roots of (2) are all real and separated by those of (3) (implying that $p = 1$). These are in turn equivalent to the conditions that c_0, d_0 shall be positive, $p = q = 1$, and the roots of (3) real and separated by those of (4), and so on.

Hence, a set of necessary and sufficient conditions is

$$p = q = \cdots = 1;\ c_0,\ d_0,\ \cdots,\ \text{all positive.}$$

It may be shown that these are equivalent to the conditions

$$\begin{vmatrix} 1 & b_1 & b_2 \\ 1 & a_1 & a_2 \\ 0 & 1 & b_1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & b_1 & b_2 & b_3 & b_4 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & b_1 & b_2 \end{vmatrix}, \qquad \cdots.$$

all positive. For the calculation of the remainders see papers by E. B. Van Vleck (*Annals of Mathematics*, 1899, pp. 1–13) and A. J. Pell and R. L. Gordon (*Annals of Mathematics*, June, 1917, pp. 188–193).

2701 [May, 1918]. Proposed by E. H. WORTHINGTON, University of Pennsylvania.

Find the sum of the infinite series

$$\frac{1}{5} + \frac{1\cdot 2}{5\cdot 7}r + \frac{1\cdot 2\cdot 3}{5\cdot 7\cdot 9}r^2 + \cdots + \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}r^{n-1} + \cdots$$

Verify your result for $r = 0$ and $r = 1$.

SOLUTION BY A. M. HARDING, University of Arkansas.

It can be easily shown that the following equation holds for all integral values of $n \geqq 0$:

$$\int_0^1 t^{1/2}(1-t)^{n+1}dt = \frac{2(n+1)}{2n+5}\int_0^1 t^{1/2}(1-t)^n dt.$$

Multiplying both sides by x^n and substituting $n = 0.\ 1, 2, \cdots (n-1)$, gives

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)dt = \tfrac{1}{5},$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^2 x dt = \frac{1\cdot 2}{5\cdot 7}2x,$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^3 x^2 dt = \frac{1\cdot 2\cdot 3}{5\cdot 7\cdot 9}(2x)^2,$$

. . . .

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^n x^{n-1} dt = \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}(2x)^{n-1}.$$

. . . .

Letting $x = r/2$ and adding, we have

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)[1 + (1-t)x + \cdots + (1-t)^{n-1}x^{n-1} + \cdots]dt$$
$$= \frac{1}{5} + \frac{1\cdot 2}{5\cdot 7}r + \cdots + \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}r^{n-1} + \cdots.$$

The series in the right member of this equation converges if $r < 2$. In this case, the series under the integral converges uniformly to the sum $\dfrac{1}{1-(1-t)r/2}$. Hence, the sum of the given series is given by

$$S = \frac{3}{2}\int_0^1 \frac{t^{1/2}(1-\mathrm{t})}{2-(1-t)r}\,dt.$$

When $r = 0$, we find $S = 1/5$, and when $r = 1$, the integral reduces to $5 - (3\pi/2)$; we do not know of any evaluation of the series independent of the integral calculus.

Also solved by H. F. GUMMER and E. H. CLARKE.

2709 [June, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

The following problem was suggested to the proposer by a professor of biology, who has found the result useful in certain problems concerning the equilibrium of chemical reactions. Starting with

$$\mu(c_1 + y - x)(c_2 - x)(c_3 - x)\cdots(c_n - x) = \lambda(b_1 + x)(b_2 + x)(b_3 + x)\cdots(b_m + x),$$

where $\mu c_1 c_2 c_3 \cdots c_n = \lambda b_1 b_2 b_3 \cdots b_m$ (all the letters being positive), find the limit of x/y as y approaches zero; and show that for small values of y, the value of x/y is always *less* than this limit.

I. SOLUTION BY THE PROPOSER.

Let $y = f(x) = f(0) + f'(0)\cdot x + \frac{1}{2}f''(0)\cdot x^2 + \cdots$.

First. From the given equations, when $x = 0$, $y = 0$; hence $f(0) = 0$.

Second. Taking logarithms and differentiating, we have

$$\frac{f'(x) - 1}{c_1 + y - x} = \left(\frac{1}{c_2 - x} + \frac{1}{c_3 - x} + \cdots + \frac{1}{c_n - x}\right) + \left(\frac{1}{b_1 + x} + \frac{1}{b_2 + x} + \frac{1}{b_3 + x} + \cdots + \frac{1}{b_m + x}\right),$$

whence

$$\frac{f'(0)}{c_1} = \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \cdots + \frac{1}{c_n}\right) + \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots + \frac{1}{b_m}\right).$$

Third. Differentiating again, we have

$$\frac{(c_1 + y - x)f''(x) - [f'(x) - 1]^2}{(c_1 + y - x)^2} = \left[\frac{1}{(c_2 - x)^2} + \frac{1}{(c_3 - x)^2} + \cdots + \frac{1}{(c_n - x)^2}\right]$$
$$- \left[\frac{1}{(b_1 + x)^2} + \frac{1}{(b_2 + x)^2} + \frac{1}{(b_3 + x)^2} + \cdots + \frac{1}{(b_m + x)^2}\right],$$

whence

$$\frac{f''(0)}{c_1} = \left[\left(\frac{1}{c_2} + \frac{1}{c_3} + \cdots + \frac{1}{c_n}\right) + \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots + \frac{1}{b_m}\right)\right]^2$$
$$+ \left[\frac{1}{c_2^2} + \frac{1}{c_3^2} + \cdots + \frac{1}{c_n^2}\right] - \left[\frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2} + \cdots + \frac{1}{b_m^2}\right],$$

which is clearly *positive*.

Fourth. With these values of $f(0)$, $f'(0)$, and $f''(0)$, we have

$$\frac{y}{x} = f'(0) + \frac{f''(0)}{2}\cdot x + \cdots, \qquad \text{or} \qquad \frac{x}{y} = \frac{1}{f'(0)} - \frac{f''(0)}{2[f'(0)]^2}\cdot x + \cdots,$$

from which the required solution is obvious.

Note: It is clear that x and y have like signs for small values of x since $f'(0) > 0$.

II. SOLUTION BY A. M. HARDING, University of Arkansas.

Solving the given equation for y gives

$$y = x - c_1 + \frac{\lambda(b_1 + x)(b_2 + x)\cdots(b_m + x)}{\mu(c_2 - x)(c_3 - x)\cdots(c_n - x)} = x - c_1 + \frac{c_1(1 + x/b_1)(1 + x/b_2)\cdots(1 + x/b_m)}{(1 - x/c_2)(1 - x/c_3)\cdots(1 - x/c_n)}.$$

Let c_k denote the smallest of $c_2, c_3, \cdots c_n$. Then, if $x < c_k$, each of the expressions $\dfrac{1}{1 - x/c_i}$, $(i = 2, 3, \cdots, n)$, may be expanded into a convergent series and we obtain, after reduction,